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LETTER TO THE EDITOR

Universal amplitude–exponent relations for interfaces and conformal mappings

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Abstract. We consider the finite-size (FS) scaling behaviour of the transverse correlation lengths $\xi_\parallel^{(n)}$ for an interface confined in a strip geometry exactly at a second-order fluctuation-dominated wetting transition. We show that the dimensionless amplitude ratios $A_\parallel^{(n)} = \Sigma L^2/k_B T\xi_\parallel^{(n)}$ (with Σ the surface stiffness coefficient) are related to the short-distance expansion critical exponent θ by $A_\parallel^{(n)} = \pi^2 n(n+\theta-1)/2$ for all fluctuation-dominated wetting transitions, provided the binding potential $V(y)$ in the capillary-wave model is conformally mapped from the semi-infinite plane. We argue that the existence of this amplitude-exponent relation reflects the conformal invariance of the capillary-wave model Lagrange density and list a number of one-point functions which can be shown to exhibit this invariance explicitly.

A remarkable conclusion of studies of finite-size (FS) effects at bulk criticality is the existence of universal amplitude–exponent relations [1] which unify FS scaling behaviour [2] in different universality classes. For example consider a model Hamiltonian defined on a periodic strip of infinite length but finite width L. FS scaling and hyperuniversality (see e.g. [2]) imply that the (true) correlation length ξ at bulk criticality is related to L by $L/\xi = A$ (for $L \to \infty$) with A a universal number (within a given universality class). Moreover, a number of exact analyses [1] show that for Ising, Potts, eight-vertex and other models

$$A = \pi\eta \tag{1}$$

with η the bulk critical pair correlation function exponent. Consequently the quotient A/η is superuniversal (being the same number in different universality classes). Cardy [3] has shown that (1) may be viewed as a consequence of local (conformal) scale invariance at bulk critical points.

In the present letter we show how FS scaling amplitudes for interfaces and interfacial unbinding (wetting) transitions (for a review of wetting phenomena, see e.g. [4]) in two bulk dimensions may be similarly related to wetting critical exponents [5a, b]. Furthermore, we argue that the resulting unification of FS scaling behaviour at interfaces may be regarded as a consequence of the local conformal invariance of the effective interfacial (capillary-wave) model Lagrange density.

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To begin, consider an interface, separating two 'bulk' phases, confined to a strip of infinite length ($M=\infty$) and finite width L. The thermal capillary-wave-like fluctuations of the interface are described by the continuum effective interfacial (capillary-wave) Hamiltonian (see e.g. [6] for a recent discussion)

$$H[y(x)]=\int_{-\infty}^{+\infty}\left(\frac{\Sigma}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+V(y(x))\right)\mathrm{d}x \tag{2}$$

where Σ is the surface stiffness coefficient and $V(y(x))$ is the interfacial binding potential. The single-valued graph $y(x)$ measures the instantaneous height of the interface separating the phases. We shall assume that each (semi-infinite) wall is wet by a different phase at exactly the same critical value of some field(s) (temperature or chemical potential for fluids and surface field or bulk field for Ising-like magnets). In the strip geometry the interface wanders widely between the two walls exactly at the wetting transition giving rise to a variety of intriguing FS behaviour [7–9]. Here we shall confine ourselves to studying FS effects at fluctuation-dominated continuous-wetting transitions. Further we shall assume that the wetting transitions occurring at each semi-infinite surface belong to the same fluctuation-scaling regime (for a detailed discussion of the fluctuation regimes at wetting transitions, see e.g. [10a, b]) and distinguish between FS effects at strong, weak and *intermediate fluctuation-scaling* critical wetting transitions.

The quantities we are interested in are the transverse correlation lengths $\xi_\parallel^{(n)}$ ($n=1,2,\ldots,\infty$). Here $\xi_\parallel^{(1)}$ corresponds to the true correlation length governing the asymptotic exponential decay of two-body correlations parallel to the walls. The $\xi_\parallel^{(n)}$ are easily calculated as $\xi_\parallel^{(n)}=(\beta(E_n-E_0))^{-1}$ where E_m ($m=0,1,2,\ldots,\infty$) are the eigenvalues of the Schrödinger operator [10a]

$$\left(-\frac{1}{2\beta^2\Sigma}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+V(y)-E_m\right)\psi_m(y)=0 \tag{3}$$

and $\beta=1/k_\mathrm{B}T$. From scaling theories [11, 12] we expect $\xi_\parallel^{(n)}\propto L^2$ for FS effects at fluctuation-dominated wetting transitions. It is therefore convenient to define a set of dimensionless amplitude ratios

$$A_\parallel^{(n)}\equiv\beta\Sigma L^2/\xi_\parallel^{(n)} \tag{4}$$

by analogy with the bulk critical case. We expect $A_\parallel^{(n)}$ to be universal in the strong-fluctuation (SFL) and weak-fluctuation (WFL) scaling regimes in the limit of large L. The explicit calculation of $A_\parallel^{(n)}$ within the various regimes is quite straightforward and follows from analysis described in detail elsewhere [13, 14].

In the WFL scaling regime we find

$$A_\parallel^{(n)}=\frac{\pi^2}{2}n(n+2)\qquad \text{WFL} \tag{5}$$

which demonstrates that the $A_\parallel^{(n)}$ are indeed universal. The result is independent of the precise details of the wetting transition occurring at each semi-infinite surface [13] (provided the transition belongs to the WFL regime) and is unchanged if irrelevant operators in $V(y)$ are allowed for. FS effects at WFL regime transitions include a number of interesting cases. For example the above result is equivalent to the conclusions of Privman and Svrakic [15] who studied the correlation lengths in a solid-on-solid model of an Ising strip with fixed $\pm$ boundary conditions. More recently the result for $A_{\|}^{(1)}$ has been derived by Abraham *et al* [16] for an Ising model strip with opposite surface fields. The WFL regime result (5) is pertinent to the case where the magnitudes of the surface fields are greater than the value at which critical wetting occurs in the limit $L\to\infty$. WFL regime amplitudes (5) describe generic FS effects at fluctuation-dominated complete-wetting transitions; e.g. suppose a two-dimensional binary liquid mixture (below its consolute point) is confined between two walls each of which is completely wet by different phases. Recall that dispersion forces are irrelevant for complete wetting in $d=2$ so that the transition is fluctuation-dominated [17]. Consequently, the WFL regime amplitudes (5) (with Σ the *surface tension* of the liquid–liquid interface) also describe this case.

The FS amplitudes $A_{\|}^{(n)}$ may also be calculated in the SFL scaling regime. We find

$$A_{\|}^{(n)}=\frac{\pi^2 n^2}{2} \qquad \text{SFL} \tag{6}$$

for this case. As for the WFL regime the result (6) is not altered by allowing for long-ranged irrelevant operators in $V(y)$. It is interesting to note that the result (6) is equivalent to the conclusions of Privman and Svrakic [15] for the correlation lengths in the Ising strip with anti-periodic (AP) boundary conditions. The reason for the identity of these two results is not at present clear. The result (6) (at least for $n=1$) should be amenable to test in the analytic work of Abraham *et al* [16] on the Ising strip. The SFL regime amplitudes (6) describe the FS effects in the Ising strip when the (opposite) surface fields take their critical wetting value [12].

At present the two sets of amplitudes (5) and (6) seem unrelated. Further insight into their structure arises if we allow for marginal long-ranged forces in $V(y)$. This enables us to study $A_{\|}^{(n)}$ at the WFL/MF (mean-field) boundary and the SFL/WFL boundary which constitute the intermediate fluctuation (IFL) scaling regimes. Recall that at a wetting transition the IFL regime describes the case where $V(y)$ contains a term $V(y)\sim wy^{-2}$ $(y\to\infty)$ exactly at the transition. In the strip geometry we follow our earlier analysis [13, 18] and set

$$V(y)=\frac{w\pi^2}{L^2}\operatorname{cosec}^2\frac{\pi y}{L} \qquad 0<y<L \tag{7}$$

and distinguish the two IFL regimes by means of suitable boundary conditions [13]. Note that in the limit $L\to\infty$ (7) recovers a pure power-law decay. For the WFL/MF case it is necessary to restrict $w\geqslant -1(8\Sigma\beta^2)$ whilst for the SFL/WFL boundary we require $-3(8\Sigma\beta^2)>w\geqslant -1(8\Sigma\beta^2)$. This latter condition on w ensures that we only consider FS effects in subregimes (A) and (B) of the model of Lipowsky and Nieuwenhuizen [19]. We do not consider FS effects at the anomalous subregime (C) in the present paper. For the potential (7) an analytic solution for $A_{\|}^{(n)}$ is possible [13]. For the WFL/MF case we find

$$A_{\|}^{(n)}=\frac{\pi^2 n(n+1+\sqrt{1+8w\Sigma\beta^2})}{2} \qquad \text{WFL/MF} \tag{8}$$

whilst for the SFL/WFL analysis shows that

$$A_{\|}^{(n)}=\frac{\pi^2}{2}n(n+\sqrt{1+8w\Sigma\beta^2}-1) \qquad \text{SFL/WFL}. \tag{9}$$

Setting $w=0$ in (8) and (9) reproduces the WFL and SFL result respectively. A remarkable feature of the results (5), (6), (8) and (9) is that they all satisfy the amplitude–exponent relations

$$A_{\|}^{(n)}=\frac{\pi^2}{2}n(n+\theta-1) \qquad \text{all regimes} \tag{10}$$

where θ is the short-distance expansion (SDE) critical exponent which characterizes the perpendicular algebraic decay of the order-parameter profile and correlation functions at critical (and complete) wetting transitions [5a, b]: if $C(z)$ denotes the local value of the order parameter (magnetization, number density, concentration) a distance z from the wall then θ is defined from the asymptotic SDE expansion

$$\frac{C(z)-C_A}{C_A-C_B}\approx\left(\frac{z}{\xi_\perp}\right)^\theta \qquad 0<\frac{z}{\xi_\perp}\ll 1 \tag{11}$$

where C_A is the bulk value of the phase absorbed at the wall and C_B denotes the value of the order parameter at $z=\infty$. Here $\xi_\perp$ denotes the perpendicular correlation length which, recall, diverges as the wetting transition is approached. In $d=2$ θ is universal in the SFL regime ($\theta=1$) [5a] and universal in the WFL [5b] ($\theta=3$). At the WFL/MF boundary $\theta=2+\sqrt{1+8w\Sigma\beta^2}$ [5b] whilst at the SFL/WFL regime boundary $\theta=2$ for subregime A [5b] and $\theta=2-\sqrt{1+8w\Sigma\beta^2}$ for subregime B. Equation (10) is the main result of the paper. It unifies FS scaling behaviour at SFL, WFL and IFL wetting transitions and also accounts for FS effects in the Ising strip with AP boundary conditions. Recall that the case with fixed boundary condition belongs to the WFL regime. Equation (10) is directly analogous to (1), relating a FS scaling amplitude to a critical exponent governing the algebraic decay of an order parameter. It should be emphasized, however, that the incorporation of the IFL regimes in the classification (10) is only possible for the potential (7)†. This is important in order that we may understand the origin of the amplitude–exponent relation more clearly. Here we elaborate further on our earlier treatment of local scale invariance at wetting transitions [13, 18]. The fixed point Hamiltonian $H^*[y(x)]$ at a two-dimensional continuous-wetting transition is invariant with respect to a renormalization group (RG) transformation which rescales the lattice (say) anisotropically [10, 20]

$$x\to x/b_\perp^2 \qquad y\to y/b_\perp$$

where $b_\perp$ denotes the perpendicular dilation factor. The parallel dilation factor $b_\|=b_\perp^2$ reflects the value of the thermal roughness exponent in $d=2$. At a fluctuation-dominated

† Equation (9) is also valid for potentials which correspond to uniform shifts of the potential in (7). For example (9) is also applicable if

$$V(y)=\frac{w\pi^2}{L^2}\cot^2\frac{\pi y}{L} \qquad \text{since} \qquad \cot^2\phi=\operatorname{cosec}^2\phi-1.$$

wetting transition one might expect that a suitably well defined two-point function $C_{(2)}(y_1, y_2; x_{12})$ (with $x_{12}=|x_1-x_2|$) satisfies the homogeneity condition

$$C_{(2)}(y_1/b_\perp, y_2/b_\perp; x_{12}/b_\perp^2) = b_\perp^{-2\phi} C_{(2)}(y_1, y_2; x_{12}) \tag{12}$$

with ϕ the scaling dimension of the local operator in terms of which $C_{(2)}$ is defined. In contrast the corresponding one-point function $C_{(1)}(y)$ satisfies the somewhat simpler equation

$$C_{(1)}(y/b_\perp) = b_\perp^{-\phi} C_{(1)}(y) \tag{13}$$

and the relation between $b_\perp$ and $b_\parallel$ is not explicit. This clearly reflects the translational invariance of $C_{(1)}(y)$ (in the x direction) for the wetting transition in the semi-infinite geometry. To proceed consider the case where the interface is of finite length (M say) with fixed end points $y(0), y(M)$. The partition function $Z(y(0), y(M); M)$ may be expressed in spectral form as

$$Z(y(0), y(M); M) = \sum_n \psi_n(y(0))\psi_n^*(y(M))\,\mathrm{e}^{-\beta E_n M} \tag{14}$$

provided $V(y)$ is independent of x. The M independence of the eigenfunctions $\psi_n(y)$ implies that the eigenstates follow from the form of the interfacial Lagrange density:

$$L^I(y(x)) = \frac{\Sigma}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + V(y) \tag{15}$$

with M infinitesimal. Under a conformal mapping $z(\equiv x+\mathrm{i}y$ in the semi-infinite plane$) \to w(z) = u+\mathrm{i}v$ (with $w(z)$ an arbitrary analytic function which preserves the boundary $y=v=0$) the (local) structure of the free-part of the Lagrange density retains a capillary-wave form over an infinitesimal distance. To see this we write the free Lagrange density $L_0^I(y(x))$ as

$$L_0^I(y(x)) \equiv \frac{\Sigma}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{\Sigma}{2}(\arctan\Delta)^2 \tag{16}$$

where Δ is defined as the angle between the tangent line and the boundary $y=0$. Under a conformal mapping $z \to w(z)$ the tangent line becomes nonlinear but the local angle between it and the conserved boundary remains invariant, i.e. the angle Δ is the same defined in both co-ordinate systems. In this sense the capillary-wave form of L_0^I is preserved by the conformal transformation. Since the form of the Lagrange density, over an infinitesimal distance ('time'), is all that is needed to define the ('time-independent') eigenfunctions in a system with x ('time') independent $V(y(x))$ the conservation of (a) the local structure (i.e. capillary-wave form) of the Lagrange density, and (b) translational invariance, under a given mapping, implies the existence of a restricted class of one-point functions which obey the local scale-invariant generalization of (13) with an *isotropic local dilation* factor, i.e.

$$P(v) = |w'(z)|^\phi P(y) \tag{17}$$

where $P(l)$ are functions of the eigenvectors $\psi_n(l)$. For our present problem we note that the analytic function

$$w(z) = \frac{L}{\pi}\ln z \tag{18}$$

maps the semi-infinite plane geometry to the strip and satisfies conditions (a) and (b) above. To see this note that the semi-infinite potential $V(y) = wy^{-2}$ satisfies the homogeneity law $V(y)=b^{-2}V(y/b)$. The one-body functions (17) therefore correspond to an interface in the transformed geometry with $V(v)=|w'(z)|^2V(y)$. For the logarithmic mapping (18) this yields the potential (7) which is indeed translationally invariant. Under the mapping we assume that the boundary conditions on the eigenfunctions at $y=v=0$ remain the same. This allows us to distinguish the SFL/WFL and WFL/MF boundary regimes as well as the SFL and WFL scaling regimes. As mentioned earlier the latter regimes correspond to the case $w=0$.

To illustrate the above we consider some examples of the restricted class of one-point functions satisfying (17). These include the simple functions

$$\psi_0(l),\ \int_0^l \psi_1(l')\,\mathrm{d}l',\ \int_0^l \psi_0(l')\psi_1(l')\,\mathrm{d}l',\ \int_0^l \psi_0(l')(l-l')\psi_2(l')\,\mathrm{d}l'$$

as well as more complicated examples. In the semi-infinite geometry these functions exhibit (different) pure power-law SDE which can be expressed in terms of θ [14]†. For example, exactly at the wetting transition

$$\frac{\psi_0(y)}{\psi_0(a)}=\left(\frac{y}{a}\right)^{(\theta-1)/2}$$

with $0<a\ll 1$. Consequently in the strip geometry (17) and (18) imply

$$\psi_0(v)\propto\left(\sin\frac{\pi v}{L}\right)^{(\theta-1)/2}\qquad 0<v<L \tag{19}$$

for the potential (7) and for the SFL and WFL regimes. It is easy to show by substitution [18] that this is the exact ground-state wavefunction for all these regimes. Similar remarks apply to the other functions listed above which correctly identify the form of the excited state eigenfunctions in the strip. The existence of a restricted class of functions which map as (17) implies that the eigenfunctions and hence eigenvalues in the strip geometry can be universally parametrized in terms of θ [13]. This is clear from the form of $\psi_0(v)$ in equation (18). That the eigenvalues satisfy the same function of θ immediately explains why it is possible to unify the FS effects in the different regimes via amplitude–exponent relations of which (10) is an example.

In summary, we have demonstrated the existence of universal amplitude–exponent relations for wetting transitions in two dimensions and related this to the local conformal invariance of the capillary-wave model Lagrange density. This illustrates that the symmetry manifested in the local scale invariance of one-point functions can be different from the symmetry manifested by the global scale invariance of two-body operators at second-order phase transitions.

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† The excited state eigenfunctions $\psi_n(l)$ with $n>1$ are well defined for the WFL/MF problem so that the mapping of these functions from the semi-infinite plane is well behaved. For the SFL regime where there is a continuum for scattering states in the semi-infinite system the mapping (17) and (18) still formally reproduces the correct $\psi_1(l)$, $\psi_2(l)$ even though these latter functions are strictly not defined. For this problem it is better to consider the inverse logarithmic mapping of one-point functions from the strip to the semi-infinite plane since the $\psi_n(l)$ are well defined for the former geometry. Since the behaviour of $\psi_n(l)$ in the semi-infinite plane is characterized by a power-law behaviour independent of n (for asymptotically small energies) the inverse mapping cannot distinguish whether this behaviour reflects the presence of a discrete or continuous spectrum.

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